Last update: 9:28am, Mar 27.

**Math 451: Intro. to General Topology** | HOMEWORK 5 | **Due: Mar 24, 23:59**

**1.** Prove that a topological space $X$ is $T_1$ if and only if every singleton $\{x\}$ in $X$ is closed.

**2.** Let $X$ be a $T_1$ topological space. Prove that $X$ is normal (i.e. $T_4$) if and only if for every closed $A \subseteq X$ and open $W \supseteq A$, there is an open $U \subseteq X$ with $A \subseteq U \subseteq \overline{U} \subseteq W$.

**3.** Follow the steps below to prove that regular Lindelöf topological spaces are normal.

Let $X$ be a regular Lindelöf topological space and $A, B \subseteq X$ be disjoint closed sets.

(a) For each $a \in A$ there is an open set $U_a \ni a$ such that $\overline{U_a} \subseteq B^c$. Likewise, for each $b \in B$ there is an open set $V_b \ni b$ such that $\overline{V_b} \subseteq A^c$.

(b) Deduce that there are countable open covers $(U_n)_{n \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ of $A$ and $B$, respectively, such that $\overline{U_n} \subseteq B^c$ and $\overline{V_n} \subseteq A^c$.

(c) For each $n \in \mathbb{N}$, put $U'_n := U_n \setminus \bigcup_{i \leqslant n} \overline{V_i}$ and $V'_n := V_n \setminus \bigcup_{i \leqslant n} \overline{U_i}$. Show that $U := \bigcup_{n \in \mathbb{N}} U'_n$ and $V := \bigcup_{n \in \mathbb{N}} V'_n$ are disjoint open sets with $U \supseteq A$ and $V \supseteq B$.

**4.** Let $X$ be a topological space and $D \subseteq X$ be a dense set.

(a) Prove that for any open set $U \subseteq X$, the set $D \cap U$ is dense in $U$. Deduce that in fact, $D \cap U$ is dense in $\overline{U}$.

(b) Let $U, V \subseteq X$ be open sets. Prove that if $\overline{U} \neq \overline{V}$ then $D \cap \overline{U} \neq D \cap \overline{V}$.

(c) Give a counterexample for $X := \mathbb{R}$ and $D := \mathbb{Q}$ to part (b) without closures, i.e. find distinct open sets $U, V \subseteq \mathbb{R}$ with $\mathbb{Q} \cap U = \mathbb{Q} \cap V$.

**5.** [*Optional*] Follow the steps below to prove that the Sorgenfrey plane is not normal.

Recall that the Sorgenfrey plane is the space $X := \mathbb{R} \times \mathbb{R}$ with the topology $\mathcal{S}$ generated by half-open rectangles $[a, b) \times [c, d)$, where $a, b, c, d \in \mathbb{R}$. Let $\Delta := \{(x, -x) : x \in \mathbb{R} \text{ and } |x| \geqslant 1\}$. Below, all topological terminology is with respect to the Sorgenfrey topology.

(a) Show that $\Delta$ is a closed set in $X$ of cardinality continuum (i.e. equinumerous with $\mathbb{R}$).

(b) Prove that $\Delta$ is discrete in the subspace topology, i.e. every singleton $\{(x, -x)\}$ is relatively open in $\Delta$. Deduce that every subset of $\Delta$ is closed in $X$.

(c) Now suppose towards a contradiction that $X$ is normal, so for each subset $A \subseteq \Delta$, the sets $A$ and $\Delta \setminus A$ are separated by disjoint open sets. Conclude that there are $2^{\mathbb{R}}$-many open sets $U$ with pairwise distinct closures.

(d) Note that $D := \mathbb{Q}^2$ is dense in $X$ and $D$ has exactly $2^{\mathbb{N}} \equiv \mathbb{R}$-many subsets. Derive a contradiction to part (c) via Question 4(b).

**6.** Let $X, Y, Z$ be topological spaces and let $f : X \to Y$ and $g : Y \to Z$. Recall that the composition $g \circ f : X \to Z$ is defined by $x \mapsto g(f(x))$. Prove:

(a) The composition of continuous functions is continuous, i.e. if $f$ and $g$ are continuous, then so is $g \circ f$.

(b) If $f$ is continuous at $x \in X$ and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$.

**7.** [*Optional*] Prove that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Hint: Build a homeomorphism $\mathbb{R} \setminus \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ using continued fraction expansion.

**8.** Let $X, Y$ be topological spaces, $f : X \to Y$, and $x_0 \in X$. Prove:

(a) If $x_0$ is not isolated and $Y$ is Hausdorff, then the limit $\lim_{x \to x_0} f(x)$ is unique if exists.

(b) $f$ is continuous at $x_0$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

**9.** Prove:

(a) An arbitrary product of Hausdorff topological spaces is Hausdorff.

(b) A an arbitrary product of regular topological spaces is regular.

Hint: Use the rephrasing of regularity in terms of open sets.

(c) A countable product of first countable topological spaces is first countable.